Five Year Integrated M.Sc. Examination, 2021-2022
Semester - VII
Code: PH-4-7-1

Time: Four Hours | Subject: Advanced Statistical Mechanics | Full Marks: 80

Questions are of values as indicated in the margin
Answer question number **one** and any **four** from the rest

1. Answer any **four** questions:

$4 \times 5 = 20$

(a) The angular frequency of oscillation of a quantum harmonic oscillator in two dimension is ω. If it is in contact with an external reservoir at temperature T, calculate the partition function.

(b) Write the two-particle wave functions for Bosons and Fermions respectively. From Fermionic wave function deduces Pauli's exclusion principle.

(c) Calculate the density of states of relativistic gas in $3 + 1$ spacetime dimensions.

(d) Calculate the microcanonical entropy for Fermions.

(e) Calculate the thermal de-Broglie wavelength for a system of N particles confined within a volume V at equilibrium temperature T. State the relation between interparticle distance and thermal de Broglie wavelength for the applicability of quantum statistical mechanics to describe this system of particles.

(f) A gas of molecules, each of mass m, is in thermal equilibrium at the absolute temperature T. Denote the velocity of a molecule by v, its three Cartesian components by v_x, v_y, and v_z, and its speed by v. What are the following mean values:
(i) $\overline{v_y v_x^2}$ and (ii) $\overline{v^2 v_y^2}$.

2. (a) Calculate the partition function of two Bosons each of which can occupy any of the two energy levels of 0 and ϵ. Calculate the partition function if there are two Fermions instead Bosons.

   (b) Consider black body radiation in thermal equilibrium contained in a two dimensional box. Calculate the density of states for such two dimensional photon gas. Hence show that the energy density of this black body radiation varies as T^3.

   (c) The rotational energy levels of a molecule are $E_l = l(l+1)\alpha$, where α is a constant and $l = 0, 1, 2, \ldots$. Show that the contribution of the rotational motion to the Helmholtz free energy per molecules, at low temperature in a dilute gas of these molecules is $-3K_BT\exp(2\alpha/K_BT)$.

(3+2)+(3+2)+5=15

3. (a) Prove the following relation for Fermi gas

$$\frac{n\lambda^3}{g} = f_{3/2}(z)\,,$$

where n is the particle density, g is the degeneracy of energy levels, λ is the thermal wavelength ($\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$), z is the fugacity ($z = e^{\mu/k_BT}$) and $f_n(z)$ is the Fermi function defined as

$$f_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty \frac{x^{n-1}}{z^{-1}e^x+1}dx\,.$$

   (b) Assuming Sommerfeld expansion in degenerate limit, i.e.

$$\frac{n\lambda^3}{g} = f_{3/2}(z) = \frac{(\ln z)^{3/2}}{\Gamma(5/2)}\left[1+\frac{\pi^2}{6}\frac{3}{4}(\ln z)^{-2}+\ldots\right] >> 1\,,$$

prove the following relation

$$\ln z = \beta\mathcal{E}_F\left[1-\frac{\pi^2}{12}\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\ldots\right]\,,$$

where $\mathcal{E}_F$ is the Fermi energy defined as

$$\lim_{T\to 0}\ln z = \beta\mathcal{E}_F\,.$$

8+7=15

4. (a) An ideal electron gas is confined to an area A in a two-dimensional plane at temperature T. Calculate

   i. the density of states

   ii. N, the number of electrons

   iii. E_F, the Fermi energy as a function of N.

   (b) Write grand canonical partition function for Bose gas and calculate the average occupation number ($n(\epsilon_p)$) of the p-th level with energy ϵ_p. Show that the total number of bosons in a system (N) can be expressed as $N = \sum_p n(\epsilon_p)$.

   (c) Using the expression for N, derive the expression for the fraction of particles in the ground state below critical temperature.

(2+2+2)+5+4=15

5. Calculate the contribution of electron spin to its magnetic susceptibility as follows. Consider non-interacting electrons, each subject to a Hamiltonian

$$H = -\mu_0 \, \vec{\sigma}.\vec{B} \, ,$$

where, $\mu_0 = e\hbar/2m$, and the eigenvalues of $\vec{\sigma}.\vec{B}$ are $\pm B$.

   (a) Calculate the densities $n_+ = N_+/V$, and $n_- = N_-/V$, of electrons pointing parallel and anti-parallel to the field.

   (b) Obtain the expression for the magnetization $M = \mu_0(N_+ - N_-)$, and expand the result for small B.

   (c) Sketch the zero-field susceptibility $\chi(T) = \partial M/\partial B|_{B=0}$, and indicate its behaviour at low and high temperatures.

   (d) Estimate the magnitude of χ/N for a typical metal at room temperature.

4+5+3+3=15

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6. (a) Starting from generalised Ising Hamiltonian (assuming non-zero external magnetic field) in d dimensions, derive the expression of the energy under mean-field approximation.

   (b) Hence calculate the partition function and magnetization per particle. Assuming zero external field solve the transcendental equation for magnetization by graphical method. Obtain the expression of critical temperature (T_c) and clearly explain the ferro-para transition across T_c.

   (c) Draw the magnetization(m) vs temperature (T) plots for both zero external field and non-zero external field.

   (d) Using the $m-T$ plots explain second order and first order phase transitions for zero and non-zero external field respectively.

4+4+3+4=15

7. (a) Write the 1D Ising Hamiltonian **without magnetic field**. Calculate the partition function for this model using transfer matrix.

   (b) Write the 1D Ising model Hamiltonian and, and calculate the transfer matrix **in presence of an external magnetic field** (h)? Calculate the eigenvalues of the Transfer matrix for this case, and hence calculate the average magnetization ($\bar{m}$).

   (c) Draw the average magnetization ($\bar{m}$) vs external magnetic field (h) plots for different values of temperatures. Hence show that 1D Ising model does not exhibit a phase transition at finite temperature ($T \neq 0$).

(1+4)+(1+4)+(3+2)=15